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Multi-parameter deformations and multi-particle representations of the bosonic oscillator

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Abstract. In this paper, we consider a multi-parameter deformation of the bosonic oscillator algebra and determine the consistency conditions on the parameters. For a d-dimensional oscillator we find $2d$ parameters. Finally, we present the Fock representation of this oscillator.

1. Introduction

In recent years, a great deal of attention has been paid to q-deformations of the Lie group and Lie algebras which are also called quantum groups (q-groups) and algebras [1–4]. With the discovery of these deformed algebras, q-deformations of the oscillator algebra which are called q-oscillators have become a center of attention, so that after the works of Coon and collaborators [5], Kuryshkin [6], Macfarlane [7] and Biedenharn [8] on the q-oscillators, many studies [9–16] have been done in order to find whether there are other deformed oscillators with similar properties.

In this paper, we study multi-parameter deformations of the bosonic oscillator. This study is motivated by the $U(n)$ invariant n-dimensional Newton oscillator [17] which satisfies

$$\begin{aligned} a_i a_j^* - q^2 a_j^* a_i &= H\delta_{ij}, \\ a_i H &= q^2 H a_i, \\ a_i a_j &= a_j a_i, \end{aligned} \tag{1}$$

so that an additional hermitean operator H which becomes central in the $q \to 1$ limit appears in the commutation relations. In Sect. 2, first we consider the system which characterizes the n-dimensional Newton oscillator; then, we consider the two-dimensional case and we find its representation. In Sect. 3, we generalize this representation of the 2-dimensional system to a d-dimensional one.

2. Construction of the d-boson multi-parameter oscillator and its two boson representation

As a multi-parameter generalization of (1), let us consider the following system:

$$a_i a_i^* - q_i^2 a_i^* a_i = H, \tag{2}$$

$$a_i a_j = q_{ij} a_j a_i, \tag{3}$$

$$a_i a_j^* = r_{ij} a_j^* a_i, \quad i \neq j, \tag{4}$$

$$a_i H = r_i^2 H a_i, \quad \text{where } i, j = 1, 2, \ldots, d. \tag{5}$$

First, we find the relations between the real parameters q_i, q_{ij}, r_i, r_{ij} such that these generalized commutation relations are consistent. It turns out that the most straightforward way to arrive at the consistency conditions is to consider

$$a_j a_i a_i^* - q_i^2 a_j a_i^* a_i = a_j H, \tag{6}$$

which by using (3)–(5) immediately implies that

$$q_{ij}^{-1} r_{ij} H a_j = r_j^2 H a_j \ , \tag{7}$$

$$r_j^2 = \frac{r_{ij}}{q_{ij}}, \quad \text{for } i \neq j. \tag{8}$$

By replacing i and j in (3), (4) and (8), we get the following equations:

$$q_{ij} q_{ji} = 1, \tag{9}$$

$$r_{ij} = r_{ji}, \tag{10}$$

$$r_i^2 = \frac{r_{ji}}{q_{ji}} = r_{ij} q_{ij}, \tag{11}$$

respectively. In these equations i and j are not summed over. Using (8) and (11), it is straightforward to obtain

$$r_{ij} = r_i r_j. \tag{12}$$

Substituting this into (8) gives also an equation for q_{ij}:

$$q_{ij} = \frac{r_i}{r_j}, \tag{13}$$

which means that our system can be rewritten as

$$a_i a_i^* - q_i^2 a_i^* a_i = H, \tag{14}$$

$$a_i H = r_i^2 H a_i, \tag{15}$$

$$a_i a_j^* = r_i r_j a_j^* a_i, \quad i \neq j, \tag{16}$$

$$a_i a_j = \frac{r_i}{r_j} a_j a_i, \quad i \neq j, \quad \text{where} \quad i,j = 1,2,\ldots,d. \tag{17}$$

The above relations do not require any further constraints on the parameters q_i and r_i. We will show that the quasi-bosonic algebra A_d described by these commutation relations is consistent and physically meaningful by explicitly constructing its representations. For the one boson case the algebra A_1 is the same as the Fibonnaci oscillator [11]; when $r = q$ it becomes the Newton oscillator mentioned in the introduction.

For two bosons the commutation relations become

$$a_1 a_1^* - q_1^2 a_1^* a_1 = H, \tag{18}$$
$$a_2 a_2^* - q_2^2 a_2^* a_2 = H, \tag{19}$$
$$a_1 H = r_1^2 H a_1, \tag{20}$$
$$a_2 H = r_2^2 H a_2, \tag{21}$$
$$a_1 a_2^* = r_1 r_2 a_2^* a_1, \tag{22}$$
$$a_1 a_2 = \frac{r_1}{r_2} a_2 a_1. \tag{23}$$

It is clear that, since they commute, $a_1^* a_1$ and $a_2^* a_2$ have common eigenvectors. We assume that there exists a ground state $\mid 0,0\rangle$ which satisfies

$$a_i \mid 0,0\rangle = 0, \tag{24}$$

as usual. With the successive application of the creation operators a_i^* to the ground state, we can obtain the orthonormal vectors $\mid n_1, n_2\rangle$:

$$\left(a_1^*\right)^{n_1} \left(a_2^*\right)^{n_2} \mid 0,0\rangle \propto \mid n_1, n_2\rangle, \tag{25}$$

with

$$a_1 \mid n_1, n_2\rangle = \sqrt{\varepsilon^{(1)}_{n_1,n_2}} \mid n_1 - 1, n_2\rangle, \tag{26}$$
$$a_1^* \mid n_1, n_2\rangle = \sqrt{\varepsilon^{(1)}_{n_1+1,n_2}} \mid n_1 + 1, n_2\rangle. \tag{27}$$

By considering (18) and (20), we can obtain the following second order homogeneous difference equation for $\varepsilon^{(1)}$:

$$\varepsilon^{(1)}_{n_1+1,n_2} - \left(q_1^2 + r_1^2\right) \varepsilon^{(1)}_{n_1,n_2} + q_1^2 r_1^2 \varepsilon^{(1)}_{n_1-1,n_2} = 0. \tag{28}$$

When we consider $\varepsilon^{(1)}_{0,0} = 0$ as the initial condition, it is straightforward to obtain the solution of the above equation:

$$\varepsilon^{(1)}_{n_1,n_2} = A_{n_2} \left(q_1^{2n_1} - r_1^{2n_1}\right), \tag{29}$$

where A may depend on the variable n_2. With the same consideration, from (19) and (21) we can obtain $\varepsilon^{(2)}_{n_1,n_2}$:

$$\varepsilon^{(2)}_{n_1,n_2} = B_{n_1} \left(q_2^{2n_2} - r_2^{2n_2}\right). \tag{30}$$

Here, the coefficient B may depend on the variable n_1. In order to find these n_2 and n_1 dependencies of the coefficients A and B, respectively, first we consider (18) and (19) and then operate with the operators on the two sides of these equations on the state $\mid n_1, n_2\rangle$. Thus, we get

$$\varepsilon^{(1)}_{n_1+1,n_2} - q_1^2 \varepsilon^{(1)}_{n_1,n_2} = H_{n_1,n_2}, \tag{31}$$
$$\varepsilon^{(2)}_{n_1,n_2+1} - q_2^2 \varepsilon^{(2)}_{n_1,n_2} = H_{n_1,n_2}. \tag{32}$$

Since the right hand sides of the above equations are equal to each other, we can also write

$$\frac{A_{n_2}}{r_2^{2n_2}\left(q_2^2 - r_2^2\right)} = \frac{B_{n_1}}{r_1^{2n_1}\left(q_1^2 - r_1^2\right)}, \tag{33}$$

which means that both sides are independent of n_1 and n_2. Therefore, (29) and (30) can be rewritten as

$$\varepsilon^{(1)}_{n_1,n_2} = C r_2^{2n_2} \left(q_2^2 - r_2^2\right)\left(q_1^{2n_1} - r_1^{2n_1}\right), \tag{34}$$
$$\varepsilon^{(2)}_{n_1,n_2} = C r_1^{2n_1} \left(q_1^2 - r_1^2\right)\left(q_2^{2n_2} - r_2^{2n_2}\right). \tag{35}$$

By redefining the constant C, we can write the above equations as

$$\varepsilon^{(1)}_{n_1,n_2} = C r_2^{2n_2} \frac{\left(q_1^{2n_1} - r_1^{2n_1}\right)}{\left(q_1^2 - r_1^2\right)}, \tag{36}$$
$$\varepsilon^{(2)}_{n_1,n_2} = C r_1^{2n_1} \frac{\left(q_2^{2n_2} - r_2^{2n_2}\right)}{\left(q_2^2 - r_2^2\right)}, \tag{37}$$

respectively. Finally, we obtain H_{n_1,n_2} by substituting (36) into (31) or (37) into (32):

$$H_{n_1,n_2} = C r_1^{2n_1} r_2^{2n_2}, \tag{38}$$

where

$$H \mid n_1, n_2\rangle = H_{n_1,n_2} \mid n_1, n_2\rangle. \tag{39}$$

3. Representation of the multi-dimensional oscillator

The quasi-bosonic algebra A_d was defined in (14)–(17). Now, we find the multi-particle representations of this multi-parameter deformed bosonic oscillator system.

For this system, (14)–(17), it is straightforward to see that $a_i^* a_i$ for different i have common eigenvectors. Thus, the ground state and the excited states can be denoted by

$$\mid \underbrace{0,0,...,0}_{d}\rangle : \text{groundstate}, \tag{40}$$

$$a_1^* a_1 \mid \underbrace{0,0,...,0}_{d}\rangle = a_2^* a_2 \mid \underbrace{0,0,...,0}_{d}\rangle = \ldots = a_d^* a_d \mid \underbrace{0,0,...,0}_{d}\rangle = 0, \tag{41}$$

$$\left(a_1^*\right)^{n_1} \left(a_2^*\right)^{n_2} \ldots \left(a_d^*\right)^{n_d} \mid \underbrace{0,0,...,0}_{d}\rangle \propto \mid n_1, n_2, \ldots, n_d\rangle. \tag{42}$$

With this generalization, we obtain d-copies of the second order homogeneous difference equation of (28) such that

$$\varepsilon^{(i)}_{n_1,n_2,\ldots,n_i+1,\ldots,n_d} - (q_i^2 + r_i^2)\varepsilon^{(i)}_{n_1,n_2,\ldots,n_i,\ldots,n_d} + q_i^2 r_i^2 \varepsilon^{(i)}_{n_1,n_2,\ldots,n_i-1,\ldots,n_d} = 0, \qquad (43)$$

where

$$a_i \mid n_1, n_2, \ldots, n_i, \ldots, n_d\rangle = \sqrt{\varepsilon^{(i)}_{n_1,n_2,\ldots,n_i,\ldots,n_d}} \mid n_1, n_2, \ldots, n_i - 1, \ldots, n_d\rangle, \qquad (44)$$

$$a_i^* \mid n_1, n_2, \ldots, n_i, \ldots, n_d\rangle = \sqrt{\varepsilon^{(i)}_{n_1,n_2,\ldots,n_i+1,\ldots,n_d}} \mid n_1, n_2, \ldots, n_i + 1, \ldots, n_d\rangle. \qquad (45)$$

The solution of the difference equations in (43) can be found in a manner similar to the solutions in (29):

$$\varepsilon^{(i)}_{n_1,n_2,\ldots,n_i,\ldots,n_d} = A_{n_1,n_2,\ldots,n_{i-1},n_{i+1},\ldots,n_d}(q_i^{2n_i} - r_i^{2n_i}), \qquad (46)$$

where the initial conditions are set to zero. Then by considering (2) and other coefficients as in Sect. 2, we can find A. The result is given by

$$\varepsilon^{(i)}_{n_1,n_2,\ldots,n_i,\ldots,n_d} = C \prod_{j=1}^{d} r_j^{2n_j} r_i^{-2n_i} \frac{q_i^{2n_i} - r_i^{2n_i}}{q_i^2 - r_i^2}. \qquad (47)$$

By substituting any of these coefficients into the corresponding equation obtained from (2) we find

$$H_{n_1,n_2,\ldots,n_i,\ldots,n_d} = C \prod_{j=1}^{d} r_j^{2n_j}, \qquad (48)$$

so that $\varepsilon^{(i)}$ can now be expressed in terms of H:

$$\varepsilon^{(i)}_{n_1,n_2,\ldots,n_i,\ldots,n_d} = C H_{n_1,n_2,\ldots,n_i,\ldots,n_d} r_i^{-2n_i} \frac{q_i^{2n_i} - r_i^{2n_i}}{q_i^2 - r_i^2}. \qquad (49)$$

4. Conclusion

In this study, we have constructed the multi-parameter deformed bosonic oscillator system and its Fock representation. If we consider the special case of our system (14)–(17) where all $r_i = r$ and all $q_i = q$, we realize that the spectrum described by $\varepsilon^{(i)}$ bears some resemblance to the spectrum of the d-dimensional quantum group covariant Fibonnaci oscillator in [11]. An interesting question is whether a similar multi-parameter oscillator construction with fermionic degeneracy is possible. Such a generalization requires a minus sign on the right hand side of (17). However, irrespective of how (14)–(16) are modified, a consistent algebra cannot be constructed, except for the case when all r_i are equal. In this case the commutation relations $a_i a_j^* + q^2 a_j^* a_i = H\delta_{ij}$, $a_i H = q^2 H a_i$, $a_i a_j + a_j a_i = 0$, which define the fermionic counterpart of (1), are obtained. Hence, from this point of view, the difficulty of constructing a deformed fermionic oscillator [18] is once again established. On the other hand, the consistent bosonic multi-dimensional, multi-parameter oscillator (2)–(5) provides a framework for application to various bosonic physical systems.

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